Problem 1: Interaction energy of two dipoles

Let's figure out the potential energy for a pair of ideal dipoles.

(a) First, show that the energy of an ideal dipole in an electric field is given by

$$U = -\vec{p} \cdot \vec{E}\,.$$

HINT: Start with a physical dipole, with charges q and $-q$ separated by a displacement $\vec{d}$. The potential energy of the charges is $q\,V(\vec{r}+\vec{d}) - q\,V(\vec{r})$. Now write the difference between the potential at $\vec{r}$ and $\vec{r}+\vec{d}$ as an integral of $\vec{E}\cdot d\vec{\ell}$. Next, imagine that $\vec{d}$ is very small, so the integral simplifies, and replace $q\,\vec{d}$ with $\vec{p}$.

(b) In class we worked out the electric field produced by a pure dipole $\vec{p}$ located at the origin. Show that the result we found can be written in the form

$$\vec{E}_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\,\frac{1}{r^3}\left[3\,\hat{r}\,(\vec{p}\cdot\hat{r}) - \vec{p}\right],$$

An expression like this, which refers to $\vec{p}$ and $\vec{r}$ but not to a specific choice of coordinates or orientation of the axes, is said to be "coordinate independent."

Combining the two results you just derived gives us the interaction energy of two dipoles $\vec{p}_1$ and $\vec{p}_2$ separated by a displacement $\vec{\mathscr{r}}$

$$U = \frac{1}{4\pi\epsilon_0}\,\frac{1}{\mathscr{r}^3}\left[\vec{p}_1\cdot\vec{p}_2 - 3\,(\vec{p}_1\cdot\hat{\mathscr{r}})\,(\vec{p}_2\cdot\hat{\mathscr{r}})\right].$$

This expression is a good starting point for a discussion of dipole-dipole forces.

Problem 2: Spherical shell with fixed polarization

A thick spherical shell with inner radius R_i and outer radius R_o is made of dielectric material with a fixed (or "frozen-in") polarization

$$\vec{P}(\vec{r}\,) = \alpha\,r^2\,\hat{r}$$

where α is a constant and r is the distance from the center. The shell is electrically neutral – no free charge has been placed on or in the material. Find the electric field in all three regions ($r < R_i$, $R_i < r < R_o$, and $R_o < r$) using the following methods. Which approach is easier?

(a) Determine all the bound charge and find the field it produces using

$$\oint_S d\vec{a}\cdot\vec{E} = \frac{Q_{\text{enc}}}{\epsilon_0}\,.$$

(b) Keeping in mind that there is no *free* charge in this problem, use

$$\oint_S d\vec{a}\cdot\vec{D} = Q_{f,\text{enc}}$$

to determine $\vec{D}$, then use $\vec{D} = \epsilon_0\,\vec{E} + \vec{P}$ to find $\vec{E}$.

Problem 3: Field inside a long, cylindrical dielectric

A very long cylinder of linear dielectric material is placed in an otherwise uniform electric field $\vec{E}_0$. The radius of the cylinder is R, its electric susceptibility is χ_e, and its axis is *perpindicular* to $\vec{E}_0$. Find the resulting electric field inside the cylinder.

HINT: This is similar to the dielectric sphere example that we discussed in class. You will need the separation of variables solution to Laplace's equation in cylindrical polar coordinates. Let's say that the axis of the cylinder is along the z-direction. Since the cylinder is very long the potential won't depend on z, and the sum of separable solutions is [1]

$$V(s,\phi) = \sum_{k=1}^{\infty} \left[\left(a_k\, s^k + b_k\, s^{-k} \right) \cos k\phi + \left(c_k\, s^k + d_k\, s^{-k} \right) \sin k\phi \right] ,$$

where k is an integer. [2] Notice that the problem only says that the axis of the cylinder is perpendicular to $\vec{E}_0$. That means that $\vec{E}_0$ could point any direction in what we're calling the x-y plane. You'll have to decide how it's oriented when you set up the problem; it's fine if you just assume that $\vec{E}_0$ points in the x direction.

Problem 4: Bound charge on a dielectric cube

A dielectric cube of side L, centered at the origin, carries a frozen-in polarization $\vec{P} = \alpha\, r^3\, \hat{r}$, where α is a constant. Calculate the total bound charge inside the cube and on its surface, and show that they add up to zero. (You may assume that the bound charge is the same on each of the six sides of the cube, though you should think about why this is. Rather than working in spherical coordinates, try expressing $r^3\, \hat{r}$ in Cartesian coordinates first!)

[1] I encourage you to derive this for yourself – it's not hard. There can also be a constant term, a_0, or a log term, $b_0 \log s$, in the potential. Neither term is needed in this problem, so they have not been included in the expression for $V(s,\phi)$. But in other problems they could be there.

[2] Why must k be an integer? We usually arrive at a condition like that by imposing an explicit boundary condition, but that's not the case here. Think about what happens when you change ϕ by 2π.